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Current correlations in the nonlinear Schrödinger model from conformal field theory

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Abstract. We analyse the nonlinear Schrödinger model (the one-dimensional Bose gas) by means of conformal field theory in conjunction with a newly developed cutoff procedure. This makes it possible to obtain exact expressions for current correlators in the model. A number of new results are presented, including the expression for the two-point current correlator on a finite-size strip and the general n-point current correlator on the plane. All these results have the full time dependence and are presented to order $1/g$, where g is the strength of the repulsive δ-function interaction.

Conformal field theory [1] has become one of the most actively researched areas of theoretical physics in recent years. Its connections to several branches of physics and mathematics have been discovered. We will make extensive use of the predictions of conformal field theory about the finite-size effects [2] in an exactly integrable system, the repulsive δ-function Bose gas [3], also called the nonlinear Schrödinger model (NLSM). It has been known for some time that the finite-size corrections to the ground and excited state energies immediately give us the asymptotic long-distance behaviour of two- and three-point correlators. These are properties of the conformal fixed point. However, to extend this result to a systematic asymptotic long-distance expansion (or, more generally, to obtain the exact expression for correlators) requires the use of a cutoff (necessitated by irrelevant operators) and the results become cutoff dependent. Thus, it is crucial to choose the 'correct' cutoff scheme when dealing with irrelevant operators. This has become possible only recently for the NLS model [4].

The plan of this paper is as follows. First, we provide a brief introduction to the thermodynamics of the NLS model. Then, we present a summary of the novel cutoff scheme followed by the main results. We end with some conclusions and a discussion.

The NLSM is defined as a one-dimensional finite-density Bose gas with repulsive pairwise δ-function interactions. The second quantized Hamiltonian is

$$H=\int_0^L \mathrm{d}x(\partial\psi^+\partial\psi+g:\psi^+\psi\psi^+\psi:-\mu\psi^+\psi) \tag{1}$$

where μ is the chemical potential, g is the strength of the δ-function interaction and L is the size of the system. The fields ψ, ψ^+ are characterized by standard commutation relations:

$$[\psi(x),\psi^+(y)]=\delta(x-y) \qquad [\psi,\psi]=[\psi^+,\psi^+]=0. \tag{2}$$

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This model is exactly integrable and was solved by Lieb and Liniger [3]. The eigenstates for a system of N particles have the characteristic Bethe ansatz [3, 5] form

$$\psi_N(\{\lambda_j\}) = \int_0^L \prod_{i=1}^N \mathrm{d}x_i\, \mathrm{e}^{\mathrm{i}\lambda_i x_i}\left[\prod_{1\leq j\leq i\leq N}\left(\theta(x_{ji}) + \theta(x_{ij})\frac{\lambda_{ij}-\mathrm{i}g}{\lambda_{ij}+\mathrm{i}g}\right)\right]$$

$$\psi^+(x_1)\psi^+(x_2)\ldots\psi^+(x_N)|0\rangle$$

$$x_{ji} = x_j - x_i \qquad \lambda_{ji} = \lambda_j - \lambda_i \tag{3}$$

where $|0\rangle$ is the bare Fock vacuum satisfying

$$\psi(x)|0\rangle = 0. \tag{4}$$

Here, the λ_j (the momenta) are not arbitrary but are constrained by the following transcendental equations:

$$\mathrm{e}^{\mathrm{i}\lambda_j L} = -\prod_{i=1}^N \frac{\lambda_{ji}+\mathrm{i}g}{\lambda_{ji}-\mathrm{i}g}. \tag{5}$$

For this model all the roots λ_j are real and distinct.

In the thermodynamic limit ($N\rightarrow\infty$, $L\rightarrow\infty$; $N/L = D$ is constant) one can obtain an integral equation for the density of roots $\rho(\lambda)$ for the ground state

$$\rho(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda-\mu)\rho(\mu)\,\mathrm{d}\mu = \frac{1}{2\pi}$$

where

$$K(\lambda) = \frac{2g}{\lambda^2+g^2} \tag{6}$$

and q is the Fermi momentum which is implicitly defined by

$$\lambda(\pm\tfrac{1}{2}D) = \pm q. \tag{7}$$

The energy of the ground state can now be presented as

$$E_0 = \int_{-q}^{q} \mathrm{d}\lambda\, \rho(\lambda)(\lambda^2-\mu). \tag{8}$$

Of course, this analysis can be carried out for excited states as well. Let us proceed to the new cutoff scheme.

We will need some of the terminology of conformal field theory in order to summarize the new cutoff scheme.

It has been shown [6] that the long-distance asymptotic behaviour of the NLSM is related via finite-size corrections to the $c = 1$ Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m(m^2-1)}{12}\,\delta_{m+n,o}. \tag{9}$$

The complete set of critical exponents was shown to be

$$r^2 x_p + \frac{t^2}{4x_p} + m \tag{10}$$

where r, t and m are integers and x_p depends on the velocity of sound v_s and the density via

$$x_p = \frac{v_s}{8\pi D}. \tag{11}$$

This information can be determined solely from the conformal fixed-point Hamiltonian. To be more explicit, the total Hamiltonian for the system can be expressed as

$$H = H_{\mathrm{CFT}} + \sum_i a_i \int_0^L O_i(x)\,\mathrm{d}x$$

where

$$H_{\mathrm{CFT}} = \frac{v_s}{2\pi} \int_0^L \mathrm{d}x(L_{-2} + \bar{L}_{-2}) \tag{12}$$

and O_i are irrelevant operators (dimension $O_i > 2$).

As pointed out in the introduction, before proceeding to do a systematic perturbation theory in the irrelevant operators about the conformal fixed point, one needs to choose a cutoff scheme to eliminate infinities.

Since the novel cutoff scheme has been described in some detail elsewhere [4], let us content ourselves with a precis. There are two main components:

(i) The new scheme takes the Fermi sea very serously. That is, in the sum over states that occurs repeatedly in perturbation theory, hole states are allowed to exist only inside the Fermi sea and particle states only outside. This has the effect of coupling the left and right movers. To see the force of this restriction we note that in conformal field theory the right and left movers are totally disconnected and hole states can go all the way to $-\infty$ while particle states go all the way to $+\infty$.

(ii) If an intermediate state has more than one particle–hole pair, then it could be represented as a number of different conformal states, depending on which holes we choose to pair with right-moving particles and which with left. The cutoff scheme prescribes that the antisymmetric combination of all possible conformal states is to be taken as the 'true' Bethe ansatz state.

There is a third component to the scheme which prescribes the order of summation for certain divergent sums, but since they occur only in the energies and never in the non-simultaneous correlation functions, we shall not be concerned with it here. One of the results to emerge from this treatment is that irrelevant operators of every dimension affect all the terms of the asymptotic long-distance expansion of the correlators. What saves us here is the existence of another small parameter $(1/g)$. We take the unperturbed theory to be the one at $g = \infty$, which has the Hamiltonian

$$H_\infty = D \int_0^L \mathrm{d}x\left(L_{-2} + \bar{L}_{-2} + \frac{1}{3\pi D} O_2\right)$$

where

$$O_2 = \int_0^L \mathrm{d}x(L_{-2}F_{10} + \bar{L}_{-2}F_{01})$$

$$F_{10} = \frac{\mathrm{i}}{\sqrt{x_p}}\partial\phi,\ F_{01} = \frac{\mathrm{i}}{\sqrt{x_p}}\bar{\partial}\phi \qquad x_p = \tfrac{1}{4} \quad \text{(free fermions)} \tag{13}$$

and ϕ is the fundamental Gaussian field of the $c=1$ conformal theory. O_2 commutes with the conformal field theory Hamiltonian. The eigenstates of H_∞ are precisely the Bethe ansatz states at $g=\infty$ [7].

We will refrain from expressing the perturbation to this Hamiltonian $((1/g)\hat{O})$ to order $1/g$ in terms of conformal fields. Instead we will present its off-diagonal matrix element between any two Bethe ansatz states, since that is what will be used to obtain the correlators. $\hat{O}$ changes two Bethe ansatz roots of Hamiltonian (13) (say from λ_1, λ_2 to λ_3, λ_4). Then the matrix element is (up to a sign)

$$\langle\lambda_3\lambda_4|\hat{O}|\lambda_1\lambda_2\rangle=\pm\frac{4}{L}\lambda_{12}\lambda_{34}. \tag{14}$$

For the ordering $\lambda_1>\lambda_2$; $\lambda_3>\lambda_4$; $\lambda_1+\lambda_2=\lambda_3+\lambda_4$ the sign is determined by the number of roots crossed in the transition.

Another miraculous fact which is crucial in computing the current correlators is the definition of the current:

$$j=\psi^+\psi=D+\frac{1}{2\pi}(F_{10}+\bar{F}_{10})|_{g=\infty}. \tag{15}$$

We must emphasize that the current j at any g is identical up to a constant to the conformal field $1/2\pi(F_{10}+\bar{F}_{10})$ at $g=\infty$. The non-vanishing matrix elements of $j(x)$ can be described as follows. The operator $j(x)$ can change the position of at most one Bethe ansatz root:

$$\langle\lambda_2|j(x)|\lambda_1\rangle=D\delta_{\lambda_1,\lambda_2}\pm\frac{1}{L}e^{i\lambda_{21}x}. \tag{16}$$

Once again the sign in the formula above is determined by the number of roots crossed in the transition.

We are now ready to present the results.

We will exhibit two main results. First, we will present the form of the time-dependent two-point current correlator on a strip of width L and then we will go on to the general ansatz for the time-dependent N-point current correlator, this time in the infinite plane. Both these results are exact to order $1/g$. Of course, one can always obtain the two-point correlator in the plane either as the $L\to\infty$ limit of the finite-size result or, more simply, by putting $N=2$ in the general result.

The two-point current correlator on the strip is obtained by standard perturbation theory using (14) and (16):

$$\begin{aligned}\langle j(x,t)j(0,0)\rangle=&\frac{2}{L^2}\sum_{|l_1|\rangle(N_0-1)/2}\sum_{|l_2|\leqslant(N_0-1)/2}\cos\frac{2\pi(l_1-l_2)x}{L}\\&\otimes\exp\left\{it\left[\left(\frac{2\pi}{L}\right)^2\left(1-\frac{4D}{g}\right)(l_1^2-l_2^2)+\frac{16\pi^2(l_1-l_2)^2}{L^3g}\right]\right\}\\&+\frac{8}{L^3g}\sum_{|l_1|\rangle(N_0-1)/2}\sum_{|l_2|,|l_3|\leqslant(N_0-1)/2}\cos\frac{2\pi l_{12}x}{L}e^{it(2\pi/L)^2(l_1^2-l_2^2)}\left[1+l_{12}\left(\frac{1}{l_{23}}-\frac{1}{l_{13}}\right)\right]\end{aligned} \tag{17}$$

where all the sums are over integers (half-integers) for N odd (even) and in the second sum $l_2=l_3$ is omitted.

We note that the last term in the exponential of the first term disappears in the $L \to \infty$ limit. This is a finite-size effect and will make itself felt near the edges of the strip. In the $L \to \infty$ limit the sums go over into integrals and the $l_2 \neq l_3$ exclusion manifests itself as a principal value prescription.

Let us now write down the form of the general N-point current correlator:

$$\langle j(x_1, t_1) \dots j(x_N, t_N) \rangle$$

$$= \left[\det(\chi_{ij}) + \frac{1}{\pi g} \sum_{k=1}^{N} \left(2q - \frac{\partial}{\partial \alpha} \frac{\partial}{\partial x_k} \right) \det(\chi_{ij} + \alpha T_{ij}(k)) \right] \Bigg|_{\alpha=0}$$

where

$$\chi_{ij} = \delta_{ij} \frac{q}{\pi} + \int_{-\infty}^{\infty} \frac{1}{2\pi} \mathrm{d}\lambda \; \mathrm{e}^{t_{ij}\lambda^2 - \mathrm{i}x_{ij}\lambda(1+2q/\pi g)}$$

$$\times [\theta(i>j)\theta(\lambda^2 > q^2) + \theta(i<j)\theta(\lambda^2 < q^2)]$$

$$T_{ij}(k) = P \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}t_{ik}\lambda_1^2 - \mathrm{i}t_{jk}\lambda_2^2 - \mathrm{i}x_{ik}\lambda_1 + \mathrm{i}x_{jk}\lambda_2}}{\lambda_1 - \lambda_2} \frac{1}{2\pi \mathrm{i}} \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 \tag{18}$$

$$\otimes \{\theta(i \geq k)\theta(j \leq k)\theta(\lambda_1^2 > q^2)\theta(\lambda_2^2 > q^2)$$

$$+ \theta(i \leq k)\theta(j \geq k)\theta(\lambda_1^2 < q^2)\theta(\lambda_2^2 < q^2)$$

$$- \theta(i<k)\theta(j<k)\theta(\lambda_1^2 < q^2)\theta(\lambda_2^2 > q^2)$$

$$- \theta(i>k)\theta(j>k)\theta(\lambda_1^2 > q^2)\theta(\lambda_2^2 < q^2)\}. \tag{19}$$

In the above α is a dummy variable and the P in front of the integral in the definition of T_{ij} denotes a principal value prescription. All the integrals are convergent when the time separations are non-zero, but extra care must be taken for simultaneous correlators:

$$\langle : j(x_1) \dots j(x_n) : \rangle$$

$$= \left\{ 1 + \frac{1}{g} \sum_{m=1}^{n} \left[\frac{2q}{\pi} + \tfrac{1}{2} \sum_{l=1}^{n} \varepsilon(x_l - x_m) \left(\frac{\partial}{\partial x_l} - \frac{\partial}{\partial x_m} \right) - \frac{1}{\pi} \frac{\partial}{\partial \alpha} \frac{\partial}{\partial x_m} \right] \right\}$$

$$\times \det \left(\frac{\sin[q x_{jk}(1 + 2q/g\pi)]}{\pi x_{jk}(1 + 2q/g\pi)} + \alpha \tau(x_{jm}, x_{km}) \right) \Bigg|_{\alpha=0}. \tag{20}$$

Here $\varepsilon(x)$ is the sign function, and

$$\tau(x, y) = \frac{1}{2\pi \mathrm{i}} P \int_{-q}^{q} \int_{-q}^{q} \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 \frac{\mathrm{e}^{\mathrm{i}\lambda_1 x - \mathrm{i}\lambda_2 y}}{\lambda_1 - \lambda_2}.$$

The full correlator can be restored from the normal ordered one above by means of commutation relation [2]. Note that the expression (20) is similar to the one obtained by means of the quantum inverse scattering method [8]. Minor discrepancies can be attributed to typographical errors.

There seems to be no obstruction to carrying out this procedure to higher order in $1/g$. One would have to deal with operators that would change n Bethe ansatz roots at a time. The convergence of the series in $1/g$ is an interesting open question which we intend to pursue.

It would also be interesting to generalize this procedure to models with a marginally irrelevant operator, for example the XXX model and also for minimal models, where there seems to be no small parameter which would play the role of $1/g$ in the NLSM.

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